Quasinormal Modes of Dirty Black Holes

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Quasinormal mode (QNM) gravitational radiation from black holes is expected to be observed in a few years. A perturbative formula is derived for the shifts in both the real and the imaginary part of the QNM frequencies away from those of an idealized isolated black hole. The formulation provides a tool for understanding how the astrophysical environment surrounding a black hole, e.g., a massive accretion disk, affects the QNM spectrum of gravitational waves. We show, in a simple model, that the perturbed QNM spectrum can have interesting features.

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1. Introduction. The new generation of gravitational wave observatories (LIGO, VIRGO) will soon be able to probe black holes in their dynamical interactions with the astrophysical environment (e.g., matter or another black hole falling into it). Numerical simulations [1] show that the gravitational waves emitted in this process will carry a signature associated with the well-defined quasinormal mode (QNM) frequencies of the black hole, and thereby confirm its existence.

A stationary neutral black hole in an otherwise empty and asymptotically flat spacetime is a Kerr hole (Schwarzschild hole in the case of zero angular momentum) [2]. Linearized gravitational waves propagating on the Kerr or Schwarzschild background can be described by the Klein-Gordon equation [3]:

$$\left[\partial_t^2 - \partial_x^2 + V(x)\right] \phi(x,t) = 0, \tag{1}$$

where x is a radial (tortoise) coordinate, ϕ is the radial part of a combination of the metric functions representing the gravitational wave. The potential $V(x)$ describes the scattering of the gravitational waves by the background geometry. The outgoing wave boundary condition is appropriate for waves escaping to infinity, and a monochromatic solution $[\phi \propto \exp(-i\omega t)]$ is a QNM, with $\mathrm{Im}\,\omega < 0$. The QNM spectra of Kerr and Schwarzschild black holes have been extensively studied [3], and provide a template against which one can try to determine the nature of the source; for an isolated black hole, the no-hair theorem [2] implies that the QNM spectrum depends only on the mass M and the angular momentum J.

However, the black holes that are observed will not be isolated, but will be situated at the centers of galaxies, or will be surrounded by accretion disks. Therefore the observed spectra should not be matched against those of a pure Kerr or Schwarzschild hole, but to a black hole perturbed by interactions with its surrounding — a dirty black hole. So far, the perturbation of black hole QNMs has attracted little attention, partly because a perturbative formalism for the QNMs of open systems, as opposed to the normal modes (NMs) of conservative systems, has not hitherto been available. In this paper we develop such a formalism, which then opens the way to inferring the astrophysical environment of the black holes from the observed signal, beyond M and J.

Two kinds of perturbations are involved here. In the standard black hole perturbation theory [3], (1) is obtained by linearizing the metric about the Kerr or Schwarzschild background, and the time-independent eigenvalue problem (with the outgoing wave boundary condition) determines the QNM spectrum. The second type of perturbations are *the perturbations that change the background* on which the wave propagates, e.g., by the presence of an accretion disk; these are often quasi-static, and hence separable from that of the gravitational wave perturbation by the time scales involved (in a suitable gauge choice). In this paper we focus on time-independent perturbation of the background, described by (1) with a potential $V(x) = V_0(x) + \mu V_1(x)$, $|\mu| \ll 1$. Therefore we are led to study the following eigenvalue problem in powers of μ:

$$-\phi''(x) + \left[V_0(x) + \mu V_1(x)\right]\phi = \omega^2 \phi \tag{2}$$

While reminiscent of standard textbook problems, e.g., the usual Rayleigh-Schrödinger perturbation theory (RSPT), the problem here is fundamentally different: the outgoing wave condition renders the system physically nonconservative (energy escapes to infinity) and the associated operator $-d^2/dx^2 + V(x)$ non-hermitian; hermiticity underpins the usual RSPT.

The difficulty can be seen in several guises if one tries naively to transcribe the usual formulas. The first-order shift cannot be given by the usual formula $\langle\phi_0|\mu V_1|\phi_0\rangle/\langle\phi_0|\phi_0\rangle$, in obvious notation — the usual inner product leads to $\langle\phi_0|\phi_0\rangle = \int_{-\infty}^{\infty} dx \phi_0^* \phi_0 = \infty$ since a QNM ϕ_0 extends over all space (and indeed grows exponentially at infinity). Higher-order shifts are even more problematic, since the usual RSPT formula involves a

sum over intermediate eigenstates, but now the unperturbed eigenstates do not in general form a complete basis [4], at least not in the case of black holes.

2. Formulation. Our formulation generalizes the logarithmic perturbation theory (LPT) [5] to QNM systems; LPT has the property that it does not require a complete set of eigenstates. Attention is focussed on the logarithmic derivative $f(x) = \phi'(x)/\phi(x)$. From (2)

$$f'(x) + f^2(x) - [V_0(x) + \mu V_1(x)] + \omega^2 = 0 \tag{3}$$

For any ω, we define two solutions $f_\pm(\omega, x)$ by the boundary conditions $f_\pm(\omega, x) \to \pm i\omega$ as $x \to \pm\infty$. At an eigenvalue ω, $f_+(\omega, x) = f_-(\omega, x)$.

For many cases of interest, $V(x) = V_0(x) + \mu V_1(x)$ is nontrivial only in a finite domain (L_-, L_+), and is relatively simple in the asymptotic regions $(-\infty, L_-)$ and (L_+, ∞). In particular, we assume that the asymptotic regions can be solved with the outgoing wave conditions to give the logarithmic derivatives $D_\pm(\omega) = f_\pm(\omega, L_\pm)$. We then expand all quantities in powers of μ: $f \equiv f_0 + g = f_0 + \mu g_1 + \mu^2 g_2 + \cdots$; $\omega = \omega_0 + \mu\omega_1 + \cdots$; $D_\pm = D_{\pm 0} + \mu D_{\pm 1} + \cdots$.

While the details of the derivation will be given elsewhere, the central result for the nth order shift is

$$\omega_n = \frac{\langle \phi_0 | V_n | \phi_0 \rangle}{2\omega_0 \langle \phi_0 | \phi_0 \rangle} \tag{4}$$

in which we have introduced the suggestive notation

$$\begin{aligned}\langle \phi_0 | V_n | \phi_0 \rangle = &\int_{L_-}^{L_+} V_n(x) \phi_0^2(x) dx \\ &- \Delta_{+n} \phi_0^2(L_+) + \Delta_{-n} \phi_0^2(L_-)\end{aligned} \tag{5}$$

$$\begin{aligned}\langle \phi_0 | \phi_0 \rangle = &\int_{L_-}^{L_+} \phi_0^2(x) dx \\ &+ \frac{1}{2\omega_0} \left\{ D'_{+0} \phi_0^2(L_+) - D'_{-0} \phi_0^2(L_-) \right\}\end{aligned} \tag{6}$$

Here V_1 is the perturbing potential in (2), and for $n > 1$, $V_n(x) = -\sum_{i=1}^{n-1} [g_i(x) g_{n-i}(x) + \omega_i \omega_{n-i}]$, with

$$\begin{aligned}\phi_0^2(x) g_n(x) = &\left[\omega_n D'_{-0}(\omega_0) + \Delta_{-n}\right] \phi_0^2(L_-) \\ &+ \int_{L_-}^{x} dy \left[V_n(y) - 2\omega_0 \omega_n\right] \phi_0^2(y)\end{aligned} \tag{7}$$

Here $\Delta_{\pm n}$ is the nth-order part of $D_\pm(\omega) - D_{\pm 0}(\omega_0)$; explicitly $\Delta_{\pm 1} = D_{\pm 1}$, $\Delta_{\pm 2} = D_{\pm 2} + \omega_1 D'_{\pm 1} + \frac{1}{2}\omega_1^2 D''_{\pm 0}$ etc., where all $D_{\pm n}$ and their derivatives are understood to be evaluated at the unperturbed frequency ω_0. These results express the nth order correction to the eigenvalue in quadrature in terms of lower-order quantities. One can hence in principle obtain the corrections to any order. Similar to LPT for conservative systems, a sum over intermediate states is not needed.

3. Properties of the Perturbed Spectrum For Open Systems in General. The result in (4) has been written in a way formally similar to the hermitian case. The factor $2\omega_0$ occurs because the eigenvalue is ω^2 rather than ω. The numerator and the denominator in (4) are separately independent of $L_\pm$, so that they can be given physical interpretations as a generalized matrix element and a generalized norm respectively.

The generalized norm has some unusual properties [6,7]. (a) It involves ϕ_0^2 rather than $|\phi_0|^2$, and is in general complex. (b) It involves surface terms at $x = L_\pm$, though the value of the entire expression is independent of the choice of $L_\pm$. Thus, it is not a norm in the strict sense, but rather a useful bilinear map. Nevertheless, in cases where the system parameters can be tuned so that the leakage of the wavefunction approaches zero (e.g., $V_0(x)$ contains tall barriers on both sides), the generalized norm does reduce to the usual (real and positive-definite) norm for a NM.

It is useful to define a function $H(x)$ for each QNM which depends only on the original unperturbed system

$$\frac{\delta\omega}{\delta(\mu V_1(x))} \equiv H(x) = \frac{\phi_0(x)^2}{2\omega_0 \langle \phi_0 | \phi_0 \rangle} \tag{8}$$

Both the magnitude and the phase of $H(x)$ are well defined and physically significant. The magnitude implies that we can now give a precise meaning to the normalization of a QNM, even though the wavefunction diverges at infinity. The phase of H determines the phase of the first-order shift ω_1 for a real and positive localized perturbation $V_1(x)$. The phase is intriguing because it has no counterpart for a hermitian system — in that case, $H(x)$ must be real and non-negative.

The functions $H(x)$ are then convenient objects for discussing the effect of any perturbation on the QNMs of a given system. We next present some properties of $H(x)$ for the Schwarzschild black hole.

4. General Properties of the Perturbed Spectrum of a Schwarzschild Hole. Waves propagating on the exact Schwarzschild background geometry is described by (1) with the potential [3]

$$V_{Sc}(M, x) = \left(1 - \frac{2M}{r}\right) \left[\frac{l(l+1)}{r^2} + (1 - s^2)\frac{2M}{r^3}\right] \tag{9}$$

with $x = r + 2M \ln(r/2M - 1)$, where s is the spin of the field ($s = 2$ for gravitational waves).

In Fig. 1 we plot for the $s = 0, l = 1$ case the functions $H(x)$, which *depends only on the unperturbed potential.* The diagrams refer to the lowest QNMs (labeled as $j = 0, 1, \cdots, 5$). We note that for a localized perturbation $\mu V_1(x) = \mu\delta(x - x_1)$, the frequency shift ω_1 is given by $H(x_1)$, and therefore can be read out directly from the figure. Both Re $H(x)$ and Im $H(x)$ alternate in sign as $(-1)^j$ near the event horizon. The patterns are different for different values of x_1 and not simple, demonstrating

that a localized perturbation will push the QNMs along different directions in the complex frequency plane, generating a rich pattern of frequency shifts (in contrast to shifts all of the same phase in the case of the NMs of a conservative system). This implies much better prospects for extracting information about the perturbing potential from the observed shifts.

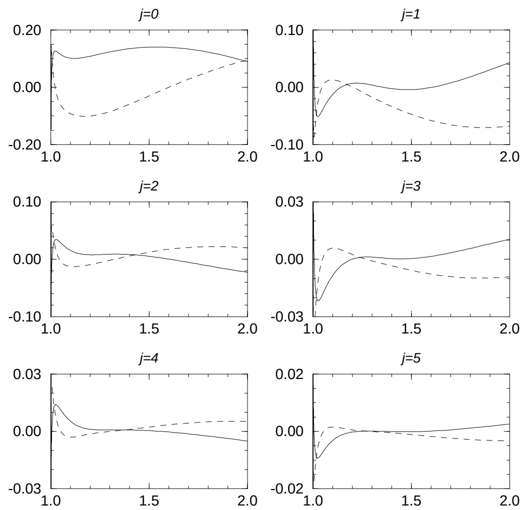

Fig. 1 Graph of Re $\left[H(x)e^{-2\gamma\sqrt{1+(x/2M)^2}}\right]$ (solid line) and Im $\left[H(x)e^{-2\gamma\sqrt{1+(x/2M)^2}}\right]$ (dashed line) vs $r/2M$, where γ=Im$(-2M\omega)$

The richness of the pattern could be diluted if the perturbation has a spatial extent Δx large compared to the typical wavelength of oscillation of $H(x)$, $\lambda \approx 2\pi/|\text{Re}\ \omega_0| \approx$ a few M. Next we discuss a model problem with an effective potential which extends over an infinite range of x.

5. Perturbed Spectrum of a Schwarzschild Black Hole in a Model Problem. Consider a Schwarzschild hole surrounded by a static shell of matter. Denote the total mass of the system as measured at infinity (ADM mass) by M_o, and the mass of the black hole as measured by its horizon surface area by M_a. The perturbation is characterized by $\mu \equiv (M_o - M_a)/M_a$ and the circumferential radius $r = r_s$ where the shell is placed. For scalar wave ($s = 0$), both the unperturbed potential V_0 and the perturbation V_1 can be given in terms of V_{Sc} in (9): $V_0(x) = V_{Sc}(M_o, x)$; $\mu V_1(x) = \kappa\delta(x - x_s) + (\beta/\alpha)V_{Sc}(M_a, x) - V_{Sc}(M_o, x)$, for $x < x_s$, and $V_1 = 0$ for $x > x_s$, where x_s is the tortoise coordinate at r_s. The constants κ, α and β are given by M_o, μ and r_s in some complicated expressions. This perturbation consists of a δ-function at the shell, plus a contribution inside the shell extending all the way to the horizon ($x \to -\infty$, $r \to 2M_a$). There is no perturbation outside the shell; in terms of the ADM mass, the outside metric is exactly that of a Schwarzschild hole with M_o.

For $x < 0$, the full potential $V = V_0 + \mu V_1$ can be expressed as a sum of exponentials, for which (2) with the outgoing wave boundary condition can be integrated analytically, thus giving the log derivatives D_-, whereas the log derivative D_+ is trivial because the perturbation vanishes outside the shell. The details of the treatment of exponential potentials will be given elsewhere [8].

We first demonstrate the convergence of the perturbation results. Fig. 2 shows the magnitude of the error in the frequencies in the 0th, 1st and 2nd order results versus μ, for $l = 1$ scalar waves, compared to the exact numerical results (which can be obtained by brute force in this simple case [9].) The error of the nth order result goes as μ^{n+1}, as it should. Detailed estimation of the error and the large n behavior of the perturbation expansion will be given elsewhere.

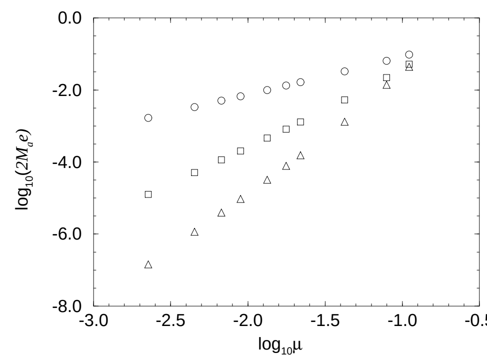

Fig. 2 The magnitude of the error e in the frequencies of the 0th (circles), 1st (squares) and 2nd order (triangles) perturbation for $l = 1$, $s = 0$, $j = 1$, and $r_s = 2.52M_a$, vs the size of the perturbation μ.

We next study the dependence on the parameters of the shell. Fig. 3a shows the trajectories of the lowest damping QNMs ($j = 0, 1, \cdots, 6$) for different r_s for the case of $l = 1$ scalar wave with $\mu = 0.01$ based on exact numerical calculation. We note the rich features of the perturbed spectra. As r_s changes, the QNMs execute complicated trajectories on the complex ω plane, with the higher-order modes moving more rapidly as r_s varies. This behavior is readily understood from the perturbation formula (4), which gives

$$\omega_1 \sim e^{2i\omega_0 x_s}/x_s^2 \qquad \text{for} \qquad x_s/2M_a \gg 1. \tag{10}$$

With Im $\omega_0 < 0$, the QNMs move away from the unperturbed positions in an exponential fashion as x_s increases, and the higher-order modes ($-$Im $2M_a\omega_0 \gg 1$) move with higher speed.

More intriguingly there are complicated fine structures in these trajectories. Fig. 3b shows the fine details of the trajectory of the $j = 0$ mode. The results obtained by direct numerical integration and by the 1st order perturbation formula are shown. For larger x_s, the trajectory shows a spiral structure, which can be explained from the first-order perturbation formula:

$$\omega_1 = \int_{-\infty}^{x_s} H(x) V_1(x) dx + \kappa(x_s) H(x_s). \qquad (11)$$

The asymptotic behavior of $H(x)$ is $H(x) \sim e^{2i\omega_0 x}$, so for large x_s, ($' = d/dx_s$)

$$\omega_1' \sim \left[V_1(x_s) + \kappa'(x_s) + 2i\omega_0 \kappa(x_s)\right] e^{2i\omega_0 x_s}. \qquad (12)$$

The exponential factor $e^{2i\omega_0 x_s}$ gives the spirial structure.

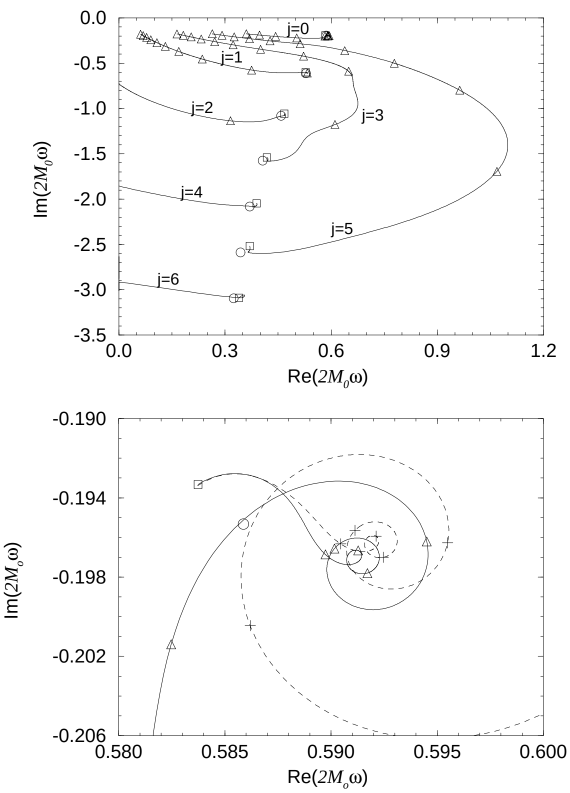

Fig. 3a (the upper graph) The trajectory of the lowest QNMs of $l = 1$ scalar waves for $\mu = 0.01$ and r_s/M_a varying from 2.22 to 60. The circles are QNMs of a bare Schwarzschild hole with mass M_o; the squares are the QNMs for $r_s = 2.22 M_a$ (The dominant energy condition is violated when $r_s < 2.22 M_a$); the triangles show the positions of QNMs at $r_s/M_a = 6$ to 60 in intervals of 6. Fig. 3b (the lower graph) shows the detail of the trajectory of the $j = 0$ mode based on exact (solid line) and 1st order (dashed line) calculation.

6. Conclusion. We have developed a formulation for the perturbation of QNMs, in close parallel to the familiar perturbation theory, which is directly applicable to black holes. With QNM gravitational wave signals from black holes to be detected soon, and many black holes expected to be perturbed by their astrophysical environments, e.g., accretion disks, this formulation will be of interest to gravitational wave astronomy.

Although the QNMs of any system can in principle be obtained through brute force numerical integration, perturbation formulas are often more revealing. We note the usefulness of perturbation theory in conventional conservative systems,e.g., in quantum mechanics. Moreover, the numerical integration of QNM spectrum is much more difficult than for NM system.

In summary, we raise the importance of studying the QNMs of dirty black holes, and have developed a perturbation formulation for this purpose. The formulation opens the way to extracting rich information from gravitational wave signals from black hole events, and leads the way to study of the inverse problem. We show in a simple example that the perturbed spectrum shows interesting features, which can be understood with the perturbation formula.

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